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Translated by M.D.F.

PMM U.S.S.R., Vol.54, No.5, pp.677-682, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

SUPERSONIC CLEAVAGE OF AN ELASTIC STRIP*

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The problem of the longitudinal cleavage of an infinite elastic strip by a thin smooth rigid wedge is examined. The wedge moves symmetrically with respect to the faces of the strip at a constant supersonic velocity. Formulas are obtained that govern the stresses in the domain of wedge contact with the elastic medium and the displacements of points of the slit edge outside the contact domain for certain relationships between the parameters of the problem. Conditions are set up for which separation of the medium from the wedge surface occurs. Unlike the case of wedge motion at a speed less than the Rayleigh velocity /1, 2/, when a crack is formed ahead of the wedge, no crack is formed when the wedge moves at supersonic speed. The contact problem of the motion of a rigid stamp with a flat smooth base at a supersonic speed over the surface of an elastic strip was investigated /3/ in a similar formulation.

1. We will first consider the auxiliary problem (plane deformation) of the motion of a concentrated force P at a constant supersonic velocity V ($V > c_1 > c_2$, where c_1 and c_2 are, respectively, the velocity of sound of longitudinal and transverse waves in the elastic medium) over the surface of an elastic strip of thickness h. Let the strip be rigidly clamped along the base. Then the boundary conditions of the auxiliary problem in a moving system of coordinates whose origin is superposed on the point of application of the concentrated force, will have the form ($\delta(x)$ is the delta-function)

$$\sigma_y = -P\delta(x),\quad \tau_{xy} = 0\ (y = 0),\ u = v = 0\ (y = -h) \tag{1.1}$$

It is well-known that such a problem reduces to finding two wave functions connected by the boundary conditions and can be solved in closed form /3/. For the system of shock waves shown in Fig.1 we present the final expression for the displacement of points of the strip upper boundary in the direction of the y-axis

$$v(x, 0) = P\Theta^{-1}[\Pi(x) - D_1\Pi(x + 2\beta h) - D_2\Pi(x + \beta h + \gamma h)] \tag{1.2}$$

$$\Pi(t) = \begin{cases} 0 & (t > 0) \\ -1 & (t < 0) \end{cases},\quad \Theta = G\frac{4\gamma(B+1)}{\gamma^2+1}$$

*Prikl. Matem. Mekhan., 54, 5, 825-830, 1990

$$D_1 = 2B\,\frac{\beta\gamma - 1}{A}, \quad D_2 = 4\,\frac{\gamma^2 - 1}{A}, \quad A = (\beta\gamma + 1)(B + 1)$$
$$B = \frac{(\gamma^2 - 1)^2}{4\beta\gamma}, \quad \beta^2 = \frac{V^2}{c_1^2} - 1, \quad \gamma^2 = \frac{V^2}{c_2^2} - 1$$

(G is the shear modulus). Formula (1.2) holds for all $x > -2\gamma h$ when $2\beta > \gamma$ as occurs in Fig.1, and also for all $x > -4\beta h$ in the case when $3\beta > \gamma > 2\beta$. For another system of shock waves shown in Fig.2, the expression for the displacements of points of the strip upper boundary in the direction of the y-axis has the form

$$v(x, 0) = P\Theta^{-1}\left[\Pi(x) - D_1\Pi(x + 2\beta h) - D_3\Pi(x + 4\beta h)\right] \tag{1.3}$$
$$D_3 = 2B(\beta\gamma - 1)^2(1 - B)/A^2$$

Formula (1.3) holds for all $x > -(\beta + \gamma)h$ when $5\beta > \gamma > 3\beta$, as well as for all $x > -6\beta h$ when $\gamma > 5\beta$ (Fig.2 corresponds to the case $4\beta > \gamma > 3\beta$).

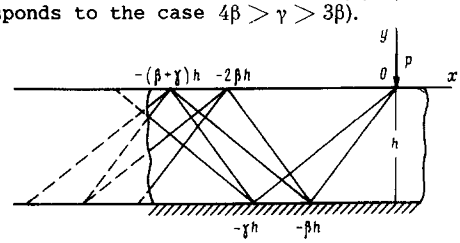

Fig.1

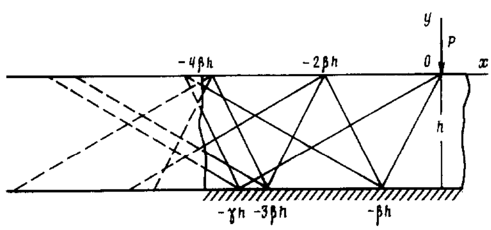

Fig.2

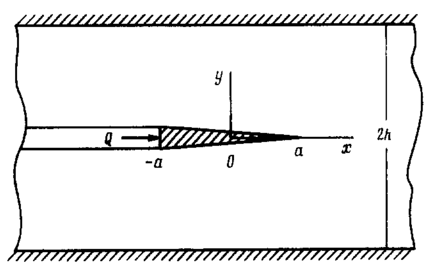

Fig.3

Let us set $V^2/c_1^2 = \sigma$, then

$$\beta = \sqrt{\sigma - 1}, \quad \gamma = \sqrt{(\sigma/\varepsilon) - 1}, \quad \varepsilon = (1 - 2\nu)\left[2(1 - \nu)\right]^{-1} < 1/2 \tag{1.4}$$

where ν is Poisson's ratio. The table gives values of the quantities γ/β, B, Θ/G, D_1, D_2, D_3 as a function of the parameter σ. It can be see that mainly $4 > \gamma/\beta > 2$, values of $\gamma/\beta > 4$ are reached for small ε (for slightly compressible materials) while values of $\gamma/\beta < 2$ are reached for ε close to $1/2$.

Using the superposition principle, we will now find the solution of the problem of the motion of a normal load $q(x)$ at supersonic speed over the surface of an elastic strip distributed over the segment $-a \leqslant x \leqslant a$. From (1.2) we have

$$v(x, 0) = \Theta^{-1}\int_{-a}^{a} q(\xi)\left[\Pi(x - \xi) - D_1\Pi(x + 2\beta h - \xi) - D_2\Pi(x + \beta h + \gamma h - \xi)\right]d\xi \tag{1.5}$$

which holds for all $x > -2\gamma h + a$, when $2\beta > \gamma$ as well as for all $x > -4\beta h + a$ when $3\beta > \gamma > 2\beta$. This can again be treated so that the relation (1.5) holds for $\lambda > \gamma^{-1}$ ($\lambda = h/a$ is the relative strip thickness), when $2\beta > \gamma$ as well as for $\lambda > (2\beta)^{-1}$, when $3\beta > \gamma > 2\beta$. Analogously, we find the relationship

$$v(x, 0) = \Theta^{-1} \int_{-a}^{a} q(\xi) [\Pi(x - \xi) - D_1 \Pi(x + 2\beta h - \xi) - D_3 \Pi(x + 4\beta h - \xi)] \, d\xi \tag{1.6}$$

from (1.3), which holds for all $x > -(\beta + \gamma) h + a$ $(\lambda > 2(\beta + \gamma)^{-1})$, when $5\beta > \gamma > 3\beta$ as well as for all $x > -6\beta h + a$ $(\lambda > (3\beta)^{-1})$, when $\gamma > 5\beta$.

ε	σ	γ/β	B	Θ/G	D_1	D_2	D_3
$^1/_4$	1	∞	∞	∞	−2	0	−2
	$1+\varepsilon$	4	2,25	5,2	0	1,846	0
	$1+2\varepsilon$	3,162	2,530	5,262	0,323	1,756	−0,0315
	$1+3\varepsilon$	2,828	2,946	5,523	0,536	1,624	−0,0950
	$1+4\varepsilon$	2,646	3,402	5,823	0,698	1,495	−0,1719
	∞	2	∞	∞	2	0	−2
$^1/_3$	1	∞	∞	∞	−2	0	−2
	$1+\varepsilon$	3	1	3,464	0	2	0
	$1+2\varepsilon$	2,450	1,378	3,805	0,279	1,916	−0,0106
	$1+3\varepsilon$	2,236	1,789	4,158	0,490	1,773	−0,0530
	$1+4\varepsilon$	2,121	2,210	4,493	0,658	1,627	−0,1184
	∞	1,732	∞	∞	2	0	−2

2. We will now formulate the fundamental problem. Let an elastic strip of thickness $2h$ clamped along the bases be cleaved by a rigid wedge of length $2a$ and apex angle 2α moving at constant velocity $V > c_1$ along the x axis (Fig.3). A driving force Q is applied to the wedge; we neglect the action of the force of friction in the area $|x| \leqslant a$ of wedge contact with the strip. By virtue of the symmetry of the problem about the x-axis we can henceforth just consider the domain $-h \leqslant y \leqslant 0$ and take the boundary conditions to the x-axis by considering the quantity $2\alpha a$ to be commensurate with the magnitude of the elastic displacements (i.e., considering the wedge to be fairly thin). Taking account of the assumptions made we will have in a moving system of coordinates connected to the wedge

$$\begin{gathered} \tau_{xy}(x, 0) = 0, \quad \sigma_y(x, 0) = 0 \quad (x < -a) \\ v(x, 0) = 0 \ (x > a), \quad v(x, 0) = -\alpha(a - x) \quad (|x| \leqslant a) \\ u(x, -h) = v(x, -h) = 0 \end{gathered} \tag{2.1}$$

We note that the third condition in (2.1) can be replaced by $\sigma_y(x, 0) = 0$ $(x > a)$, when taking into account that the wedge motion is supersonic.

We further assume that the relative strip thickness λ is so large that (1.5) and (1.6) hold. Then by satisfying the boundary conditions (2.1), we arrive in the case when $3\beta > \gamma$ at an integral equation in the contact pressure $q(\xi)$, that is the equality of the right-hand of (1.5) to the quantity $\alpha(x - a)$ $(|x| \leqslant a)$ by using (1.5). Analogously, by using (1.6) we arrive in the case when $\gamma > 3\beta$ at an integral equation that is the equality of the right-hand side of (1.6) to the quantity $\alpha(x - a)$ $(|x| \leqslant a)$.

Differentiating these integral equations with respect to x and taking into account that $\Pi'(t) = \delta(t)$, we find

$$\int_{-a}^{a} q(\xi) [\delta(x - \xi) - D_1 \delta(x + 2\beta h - \xi) - D\delta(x + \Lambda h - \xi)] \, d\xi = \Theta\alpha \quad (|x| \leqslant a) \tag{2.2}$$

$$D = D_2, \quad \Lambda = \beta + \gamma \tag{2.3}$$

$$D = D_3, \quad \Lambda = 4\beta \tag{2.4}$$

If $\lambda > \beta^{-1}$ then using the well-known properties of the delta-function we obtain an expression

$$q(x) = \Theta\alpha \tag{2.5}$$

from (2.2)-(2.4) which simultaneously satisfies the initial integral equations also. We later find the driving force from the formula

$$Q = 2\alpha \int_{-a}^{a} q(\xi) \, d\xi = 4a\Theta\alpha^2 \tag{2.6}$$

Taking account of the relationship $\int \Pi(t)\,dt = {}^1/_2(|t| - t)$, by means of (1.5), (1.6) and (2.5) we can find the width of the slit $\Gamma(x) = -2v(x, 0)$ beyond the penetrating wedge in each specific case.

From (1.5) we determine the following:

for the case $2\beta > \gamma$

$$\begin{gathered}\Gamma(x) = 2\alpha\,\{2a + E_{12}(x)\,Y_{72}(x) + D_1(x - x_1)\,Y_{21}(x)\}\ (\gamma - \beta \leqslant \lambda^{-1})\\ \Gamma(x) = 2\alpha\,\{2a + F_{12}(x)\,Y_{74}(x) + E_{12}(x)\,Y_{42}(x) +\\ D_1(x - x_1)\,Y_{21}(x)\}\ (\lambda^{-1} < \gamma - \beta < 2\lambda^{-1})\\ \Gamma(x) = 2\alpha\,\{2a - 2a\,(D_1 + D_2)\,Y_{76}(x) + F_{12}(x)\,Y_{62}(x) -\\ 2aD_1Y_{24}(x) + D_1(x - x_1)\,Y_{41}(x)\}\ (2\lambda^{-1} < \gamma - \beta)\end{gathered} \tag{2.7}$$

for the case $3\beta > \gamma > 2\beta$

$$\begin{gathered}\Gamma(x) = 2\alpha\,\{2a + F_{12}(x)\,Y_{34}(x) + E_{12}(x)\,Y_{42}(x) +\\ D_1(x - x_1)\,Y_{21}(x)\}\ (\gamma - \beta < 2\lambda^{-1})\\ \Gamma(x) = 2\alpha\,\{2a + F_{12}(x)\,Y_{32}(x) - 2aD_1Y_{24}(x) + D_1(x - x_1)\,Y_{41}(x)\}\\ (\gamma - \beta > 2\lambda^{-1},\ 3\beta - \gamma \leqslant 2\lambda^{-1})\\ \Gamma(x) = 2\alpha\,\{2a - 2a\,(D_1 + D_2)\,Y_{36}(x) + F_{12}(x)\,Y_{62}(x) -\\ 2aD_1Y_{24}(x) + D_1(x - x_1)\,Y_{41}(x)\}\quad (3\beta - \gamma > 2\lambda^{-1})\end{gathered} \tag{2.8}$$

Here and henceforth we use the notation

$$\begin{gathered}x_1 = a - 2\beta h,\ x_2 = a - \beta h - \gamma h,\ x_3 = a - 4\beta h\\ x_4 = -a - 2\beta h,\ x_5 = a - 3\beta h - \gamma h,\ x_6 = -a - \beta h - \gamma h\\ x_7 = a - 2\gamma h,\ x_8 = -a - 4\beta h,\ x_9 = a - 6\beta h\\ Y_{ij}(x) = \Pi(x_i - x)\,\Pi(x - x_j),\quad Y_{ij}(x_i) = Y_{ij}(x_j) = 1\\ E_{12}(x) = D_1(x - x_1) + D_2(x - x_2),\quad F_{1i}(x) = -2aD_1 + D_i(x - x_i)\end{gathered} \tag{2.9}$$

Analogously, taking account of relations (2.5) we determine from (1.6)

$$\begin{gathered}\Gamma(x) = 2\alpha\,\{2a + F_{13}(x)\,Y_{23}(x) - 2aD_1Y_{34}(x) + D_1(x - x_1)\,Y_{41}(x)\}\\ (0 < \gamma - 3\beta \leqslant 2\lambda^{-1})\\ \Gamma(x) = 2\alpha\,\{2a - 2a\,(D_1 + D_3)\,\Pi(x - x_8) + F_{13}(x)\,Y_{83}(x) -\\ 2aD_1Y_{34}(x) + D_1(x - x_1)\,Y_{41}(x)\}\quad (2\lambda^{-1} < \gamma - 3\beta < 2\beta)\end{gathered} \tag{2.10}$$

Formulas (2.7), (2.8) and (2.10) can be used under appropriate constraints on x mentioned in the description of expressions (1.5) and (1.6). As follows from equalities (2.7), (2.8) and (2.10), $\Gamma(x)$ is a piecewise-linear function. For $x_1 \leqslant x \leqslant -a$ the slit width is constant and equal to $4\alpha a$. For $x < x_1$ and certain relationships between the parameters of the problem, the slit edges make contact, however, by virtue of the supersonic nature of the wedge motion this does not result in a change in the contact pressures.

3. If $\lambda < \beta^{-1}$ but $\lambda > 2(\beta + \gamma)^{-1}$ in the case (2.2) and (2.3) or $\lambda > (2\beta)^{-1}$ in the case (2.2) and (2.4), then we will have from (2.2), (2.3) and (2.2), (2.4)

$$\begin{gathered}q(x) = \Theta\alpha\quad (x_1 \leqslant x \leqslant a)\\ q(x) - D_1 q(x + 2\beta h) = \Theta\alpha\quad (-a \leqslant x < x_1)\end{gathered} \tag{3.1}$$

Taking account of the first relation in (3.1) it follows from the condition $x_1 < -x_1 \leqslant x + 2\beta h \leqslant a$ that

$$q(x + 2\beta h) = \Theta\alpha\quad (-a \leqslant x < x_1) \tag{3.2}$$

We obtain from (3.1) and (3.2)

$$q(x) = \Theta\alpha \times \begin{cases}1 & (x_1 \leqslant x \leqslant 0)\\ (1 + D_1) & (-a \leqslant x < x_1)\end{cases} \tag{3.3}$$

We note that the second relation in (3.1) was erroneously taken as a difference equation in /3/ (Sect.4, Ch.5). The result (3.3) must be used in place of (4.41)-(4.48) in /3/.

Formula (3.3) shows that the contact pressure $q(x)$ is positive everywhere in the contact domain $|x| \leqslant a$ for $1 + D_1 > 0$, while it has a jump at the point $x = x_1$ for $D_1 \neq 0$. We now find the driving force Q from (2.6) and the slit width from (1.5) and (1.6). Making the calculations, we will have, respectively,

$$Q = 4a\Theta\alpha^2 [1 + D_1 (1 - \beta\lambda)] \tag{3.4}$$

$$\Gamma(x) = 2\alpha \times \begin{cases} [2a + D_1 (a + x) + D_2 (x - x_2)] & (x \leqslant x_2) \\ [2a + D_1 (a + x)] & (x_2 < x \leqslant -a) \end{cases} \tag{3.5}$$

$$\Gamma(x) = 2\alpha \times \begin{cases} [2a - 2\beta h D_1 + 2D_1^2 (\beta h - a) + D_3 (x - x_3)] & (x \leqslant x_4) \\ [2a + D_1 (a + x) + (D_1^2 + D_3)(x - x_3)] & (x_4 < x \leqslant x_3) \\ [2a + D_1 (a + x)] & (x_3 < x \leqslant -a) \end{cases} \tag{3.6}$$

Separation of the medium from the wedge at the point $x = x_1$ sets in for $1 + D_1 \leqslant 0$ (the quantity $1 + D_1$ vanishes, for example, for $\varepsilon = 1/4$, $\sigma \approx 1.0189$ and $\varepsilon = 1/3$, $\sigma \approx 1.0092$). Changing the boundary conditions for the case $1 + D_1 \leqslant 0$ and performing appropriate calculations, we find

$$q(x) = \Theta\alpha \quad (-a < x_1 \leqslant x \leqslant a), \quad Q = 4a\Theta\alpha^2\lambda\beta \tag{3.7}$$

$$\Gamma(x) = 2\alpha [2\beta h + D_1 (x - x_1) + D_2 (x - x_2) \Pi (x - x_2)] \; (x \leqslant x_1) \tag{3.8}$$

$$\Gamma(x) = 2\alpha \{2\beta h (1 - D_1) + (x - x_3) [D_1 \Pi (x_3 - x) + D_3 \Pi (x - x_3)]\} \quad (x \leqslant x_1) \tag{3.9}$$

The constraints on x mentioned in describing expressions (1.5) and (1.6) must be taken into account in (3.5) and (3.8), obtained from (1.5), and in (3.6) and (3.9) obtained from (1.6).

If the quantity $\lambda < (2\beta)^{-1}$ in the case (2.2), (2.4), but $\lambda > 2(\beta + \gamma)^{-1}$, when $5\beta > \gamma > 3\beta$ or $\lambda > (3\beta)^{-1}$ when $\gamma > 5\beta$, we will have

$$\begin{gathered} q(x) = \Theta\alpha \quad (x_1 \leqslant x \leqslant a) \\ q(x) - D_1 q(x + 2\beta h) = \Theta\alpha \quad (x_3 \leqslant x < x_1) \\ q(x) - D_1 q(x + 2\beta h) - D_3 q(x + 4\beta h) = \Theta\alpha \quad (-a \leqslant x < x_3) \end{gathered} \tag{3.10}$$

Taking account of the values

$$\begin{gathered} q(x + 2\beta h) = \Theta\alpha \times \begin{cases} 1 & (x_3 \leqslant x < x_1), \\ (1 + D_1) & (-a \leqslant x < x_3) \end{cases} \\ q(x + 4\beta h) = \Theta\alpha \quad (-a \leqslant x < x_3) \end{gathered} \tag{3.11}$$

we determine from (3.10)

$$q(x) = \Theta\alpha \times \begin{cases} 1 & (x_1 \leqslant x \leqslant a) \\ (1 + D_1) & (x_3 \leqslant x \leqslant x_1) \\ [D_1 (1 + D_1) + 1 + D_3] & (-a \leqslant x < x_3) \end{cases} \tag{3.12}$$

If the quantity $\lambda < 2(\beta + \gamma)^{-1}$ in the case (2.2), (2.3) but $\lambda > \gamma^{-1}$ when $2\beta > \gamma$ or $\lambda > (2\beta)^{-1}$ when $3\beta > \gamma > 2\beta$ then we will have a relationship different from (3.10) by replacing x_3 by x_2 and $D_3 q(x + 4\beta h)$ by $D_2 q(x + \beta h + \gamma h)$.

By analogy with the preceding, we then obtain

$$q(x) = \Theta\alpha \times \begin{cases} 1 & (x_1 \leqslant x \leqslant a) \\ (1 + D_1) & (x_2 \leqslant x \leqslant x_1) \\ (1 + D_1 + D_2) & (-a \leqslant x < x_2) \end{cases} \tag{3.13}$$

It is seen from (3.12) and (3.13) that the contact pressure $q(x)$ is positive everywhere in the contact domain $|x| \leqslant a$ if $1 + D_1 > 0$ and, respectively, $1 + D_1 + D_3 > -D_1^2$ or $1 + D_1 + D_2 > 0$. As follows from the tables, these conditions are always satisfied for $\sigma \geqslant 1 + \varepsilon$. As before, expressions can be obtained for the cases (3.12) and (3.13) that determine the driving force Q and the slit width $\Gamma(x)$, and an investigation can be made of the conditions for an elastic medium to separate from the wedge.

We note that in the case of apex angles 2α that are not too small for a penetrating rigid wedge, it would be necessary to write the fourth boundary condition in (2.1) in the form $v(x, 0) = -\sin\alpha (a - x)$ $(|x| \leqslant a)$. Then expanding the sine in series and keeping terms of order α^3, formulas for the contact pressure $q(x)$ can be obtained by the scheme elucidated above by analytic means to terms of order α^3 and for the driving force Q to terms of order α^1. The principal terms in the expansion of Q in terms of α without taking account of friction forces here would remain as before, namely, of order α^2.

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Translated by M.D.F.